# Linear stability analysis of high flow velocity jets stressed by electric fields

G.Artana[*1,2], H.Romat[*1], G.Touchard[*1]

[*1] Laboratoire de Physique et Mècanique des Fluides, Université de Poitiers, 40 Av.Recteur Pineau, 86022, POITIERS, FRANCE.
[*2] Departamento de Mecánica Aplicada, Facultad de Ingeniería, Universidad de Buenos Aires, BUENOS AIRES, ARGENTINA.

**ABSTRACT**: In this article we develop a linear stability analysis in the case of a high speed and non viscous cylindrical liquid jet subjected to an electric field. Two extreme electrical situations are considered depending on whether the surface of the liquid is equipotential or not. In each case we give the theoretical results and we conclude with a discussion on the influence of the electric field on the change of the spray angle and on the size of the droplets.

## 1. Description of the problem

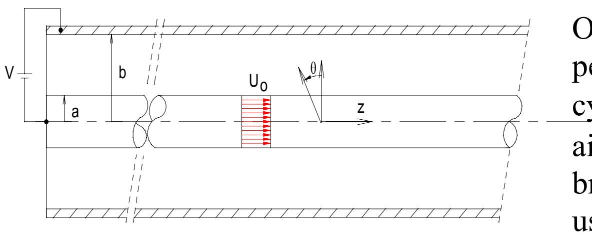


Figure N°1

On figure n°1 we represent the non perturbated situation: it is an infinite liquid cylinder flowing at uniform velocity $\vec{\mathbf{U}}_\mathbf{0}$ in air at rest, and inside a coaxial electrode brought to a different electric potential. We use a cylindrical system of coordinates moving with velocity $\vec{\mathbf{U}}_\mathbf{0}$.

In our study we neglect the viscous effects, the mass transfer between phases and the gravitational and magnetic effects. Both fluids are considered incompressible and the liquid jet velocity is considered uniform in the cross section

## 2. General Equations.

The mass conservation leads to:

**-1)** $\vec{\nabla}.\vec{\mathbf{U}}_\mathbf{i} = \mathbf{0}$ in each phase (with **i=1** for the liquid and **i=2** for the gas). Because of the absence of mass transfer between the phases, we have $(\vec{\mathbf{U}}_\mathbf{1} - \vec{\mathbf{U}}_\mathbf{2}).\vec{\mathbf{n}} = \mathbf{0}$, $\vec{\mathbf{n}}$ being the normal to the interface and $\vec{\mathbf{U}}_\mathbf{i}$ the velocity of the phase **i**.

**-2)** $\underline{\mathbf{div}_\mathbf{s}\vec{\mathbf{V}}_\zeta = \mathbf{0}}$ at the interface. For the same reason $(\vec{\mathbf{U}}_\mathrm{i} - \vec{\mathbf{V}}_\zeta).\vec{\mathbf{n}} = \mathbf{0}$. $\vec{\mathbf{V}}_\zeta$ is the velocity vector of the interface and $\mathbf{div}_\mathbf{s}$ the surfacic divergence.

As for momentum conservation, it leads to:

**-1)** $\rho_i \dfrac{\mathbf{d}\vec{\mathbf{U}}_\mathrm{i}}{\mathbf{dt}} - \vec{\nabla} \circ \vec{\vec{\mathbf{T}}}_\mathrm{i} = \vec{\mathbf{0}}$ in each phase. $\rho_\mathbf{i}$ and $\vec{\vec{\mathbf{T}}}_\mathbf{i}$ are the mass density and Maxwell's constraint tensor of the phase **i**. The expression of this tensor is: $\vec{\vec{\mathbf{T}}}_\mathrm{i} = -\mathbf{p}_\mathrm{i}\vec{\vec{\mathbf{I}}} + \vec{\vec{\mathbf{T}}}_\mathrm{i}^{\mathbf{el}}$, where $\mathbf{p_i}$ is the static pressure and $\vec{\vec{\mathbf{I}}}$ the identity tensor [1]. The components of $\vec{\vec{\mathbf{T}}}_\mathbf{i}^{\mathbf{el}}$, using Einstein's notation, are:

$(T_i^{el})_{jk} = \varepsilon_i (E_i)_k (E_i)_j - \frac{\varepsilon_i}{2}\delta_{kj}(E_i)_m (E_i)_m$, with $\varepsilon_i$ and $(E_i)_j$ the permittivity and the electric field in the **jth** direction, of the phase **i**.

-2) $-\text{div}_s(\vec{\vec{T}}_\zeta) - (\vec{\vec{T}}_1 - \vec{\vec{T}}_2)\circ \vec{n} = \vec{\vec{0}}$ at the interface. $\vec{\vec{T}}_\zeta$ is the constraint tensor in this region. Its expression is given by $\vec{\vec{T}}_\zeta = \gamma_{i-j}(\vec{\vec{I}} - \vec{n}\otimes\vec{n})$. According to [2] $\text{div}_s\, \vec{\vec{T}}_\xi = \gamma_{i-j} < \upsilon_n > \vec{n} + \text{grad}_s \gamma_{i-j}$, $<\upsilon_n>$ being the averaged surface curvature and $\gamma_{i-j}$ the surface tension. As $\gamma_{i-j}$ is constant, the surfacic gradient (**grad$_s$**) of this magnitude vanishes.

## 3. Linear Perturbation Analysis

Let us modify the equilibrium state of the surface of the jet with an infinitesimal perturbation $\eta$ Then, the radial coordinate of the perturbated surface is:

$r_s = a + \eta = a + \eta_0 \exp(i(kz + n\theta) + \omega\, t)$,

where **n** is an integer.

The unitary vector normal to the interface and the average surface curvature are: $\vec{n} = \vec{i}_r - i\frac{n\eta}{a}\vec{i}_\theta - ik\eta\,\vec{i}_z$ and $<\upsilon_n> = \vec{\nabla}.\vec{n} = \frac{1}{r_s} - \frac{1}{r_s^2}\frac{\partial^2\eta}{\partial\theta^2} - \frac{\partial^2\eta}{\partial z^2}$. Since we develop a linear analysis, we neglect all the terms involving $\eta$ to a power greater than unity.

### *3.1. Velocity fields after the perturbation*

In each phase, according to [3], the fluid flows are potential which can be expressed by $\vec{U}_i = \vec{\nabla}\varphi_i$ and $\vec{\nabla}^2\varphi_i = 0$. At $r = r_s$ , both phases must satisfy $(\vec{U}_i - \vec{V}_\zeta)\,.\vec{n} = 0$.

-1) In the gaseous phase and for **r=b** we have $\vec{U}_2 . \vec{i}_r = 0$. The velocity field, which is solution of Laplace's equation with the above boundary conditions, is given by

$$\vec{U}_2 = -U_o\,\vec{i}_z + \frac{(\omega - iU_o k)}{k}\cdot\vec{\nabla}\left(\left(\frac{I_n(kr)}{I'_n(ka)(1-\lambda)} + \frac{K_n(kr)}{K'_n(ka)(1-\frac{1}{\lambda})}\right)\eta\right),$$

in which $\lambda = \frac{I'_n(kb)\,K'_n(ka)}{I'_n(ka)\,K'_n(kb)}$ and where $I_n$ and $K_n$ are the modified Bessel functions of first and second kind, $I'_n$ and $K'_n$ are the first derivatives of these Bessel function with respect to the argument. In the equilibrium configuration the velocity field verifies the relation: $\vec{U}_2 = -U_0\vec{i}_z$.

-2) In the liquid phase, the velocity field which satisfies Laplace's equation with the boundary conditions is $\vec{U}_1 = \frac{\omega}{k\,I'_n(ka)}\vec{\nabla}(I_n(kr)\eta)$. In the non perturbated state it is naught.

### *3.2. Electric fields after the perturbation*

We analyse two different extreme situations. The first one considers that the liquid surface is equipotential. This hypothesis corresponds to a very high conductive liquid (resistivity $\rho_{el} \to 0$) and to an electric relaxation time $\tau = \rho_{el}\,\varepsilon$ inferior to the inverse of the frequency of the perturbation **(ω)**. In the second situation the jet surface after the perturbation is not an equipotential and we analyse a special condition that we name total equicharged. In this case we still consider the conductivity of the liquid high but now, τ is greater than the inverse of the frequency.

Considering Maxwell's equations in both fluids, we can express the electric field as: $\vec{\mathbf{E}} = \vec{\nabla}\phi$, then $\vec{\nabla}^2\phi = \mathbf{0}$ [1].

*3.2.1. Equipotential liquid surface*

-1) In the gaseous phase, the boundary conditions on the electric potential function $\phi$ are: $\phi(\mathbf{r_s}) = \mathbf{0}$ and $\phi(\mathbf{b}) = \mathbf{V}$. The normal electric field $\mathbf{E_n}$ satisfying the boundary conditions and Laplace's equation is:

$$\mathbf{E_n} = \frac{\mathbf{V}}{\mathbf{ln(b/a)}}\left[\frac{1}{\mathbf{r}} - \frac{\mathbf{k}}{\mathbf{a}}\left(\frac{\mathbf{I'_n(kr)}}{\mathbf{I_n(ka)(1-\beta)}} + \frac{\mathbf{K'_n(kr)}}{\mathbf{K_n(ka)(1-\frac{1}{\beta})}}\right)\right]$$

, with

$$\beta = \frac{\mathbf{K_n(ka)I_n(kb)}}{\mathbf{I_n(ka)K_n(kb)}}$$

-2) In the liquid phase we consider that there is no free charge in the bulk of the liquid, which gives $\mathbf{E_n=0}$. As the jet surface is an equipotential, the tangential electric field vanishes in both phases.

*3.2.2. Non equipotential liquid surface (total equicharded case).*

In this paragraph, we propose the following expression $\sigma = \sigma_0\, \mathbf{a/r_s}$, for the surface charge density σ of the perturbated jet, $\sigma_0$ being the one of the non perturbated state. This means that the initial total surface charge remains constant after the perturbation and that a single charge does not move neither in $\vec{\mathbf{i}}_z$ nor in $\vec{\mathbf{i}}_\theta$ direction. If we accept that there is no charge in the liquid bulk, then Gauss law leads to $\mathbf{E_n} = \mathbf{E_{on}\, a/r_s}$, $\mathbf{E_{on}}$ being the normal electric field of the non perturbated system.

**-1)** In the gaseous phase the boundary conditions are $\vec{\nabla}\phi(\mathbf{r_s})\cdot\vec{\mathbf{n}} = \mathbf{E_n(r_s)} = \frac{\mathbf{V}}{\mathbf{ln(b/a)r_s}}$ and $\phi(\mathbf{b}) = \mathbf{V}$. Therefore, the electric field is

$$\vec{\mathbf{E}} = \frac{\mathbf{V}}{\mathbf{ln(b/a)}}\vec{\nabla}\,\mathbf{ln(r/a)}$$

**-2)** In the liquid phase, since there is no charge out of the interface, the intensity of the field in the direction of the normal to the perturbated surface is equal to zero: $\mathbf{E_n=0}$.

*3.3. General equation*

The substitution of the electric and velocity fields obtained in the perturbated system, inside the equations of momentum conservation, leads to an expression which has terms independent of the perturbation and others dependent on it. The sum of all independent terms is zero, as they correspond to the equilibrium state before the perturbation. After simplification, we find the following equation which takes into account the linear terms in $\eta$:

$$\frac{\gamma_{i-j}}{a^2}(1-n^2-(ak)^2)-\rho_1\frac{I_n(ak)}{I'_n(ak)}\frac{\omega^2}{k}+\rho_2\frac{(\omega-iU_0k)^2}{k}\alpha-\varepsilon_0\frac{V^2}{a^2\ln^2(b/a)}(\frac{1}{a}+k\varsigma)=0$$

(1) with $\alpha=\frac{I_n(ka)}{I'_n(ka)(1-\lambda)}+\frac{K_n(ka)}{K'_n(ka)(1-1/\lambda)}$

The parameter $\varsigma$ depends on the electrical situation considered. In the total equicharged case it is null. If the surface is equipotential it is then : $\varsigma=\frac{I'_n(ka)}{I_n(ka)(1-\beta)}+\frac{K'_n(ka)}{K_n(ka)(1-1/\beta)}$. In this last case, we can notice that $\varsigma<0$ because $\beta>1$.

Expression (1), when air density is null, agrees with Melcher's expression for electrified jets [4], and with Levich's one, in the axilsymetric case, for non electrified jets with high flow velocities[5].

**4. Discussion**

The solution of the equation of momentum conservation at the interface gives the set of wave numbers **k** and frequencies $\omega$ that furthers instabilities by amplifying the amplitude of the perturbation. In this article, we analyse the problem of the evolution in time of a perturbation given in the space at t = 0 (temporal analysis). In that analysis the set of parameters $(\omega,k)$ of the unstable waves is obtained considering $\omega$ complex and **k** real, the amplitude of the perturbation growing with time if $\omega_r$, the real part of the frequency, is positive.

When solving complex equation (1), with a velocity $U_0$ high enough, we obtain values of **ak** greater than one. Still considering equation (1), and for those values of **ak**, one can deduce that the surface tension has a stabilising role (it diminishes $\omega_r$), and that, on the contrary, the effect of the surrounding atmosphere and the electric field for $\varsigma<0$, is destabilising (it increases $\omega_r$). The instability is convective (swept by the flow from its original location) and it moves with a velocity of group $V=\frac{\partial\omega}{\partial k}=\frac{\partial\omega_{i\,mx}}{\partial k}\approx U_0\vec{i}_z$ [3], observed from a fixed laboratory referential ($\omega_{i_{mx}}$ is the complex part of the frequency corresponding to the maximum of the growing rate $\omega_{r_{mx}}$).

On figure n°2 and figure n°3 we show a typical result of $\omega(k)$ of a high velocity jet and for the first mode. If we increase the velocity, other results show that the maximum of the growing rate also increases. With velocities not extremely high **($U_0$<150m/s)**, we observe that the electric field changes the $\omega(k)$ curves and the value and position of the maximum of $\omega_r$. For extremely high velocities aerodynamic effect is much more important than electric effect. The results we obtain with higher modes have always lower $\omega_{rmx}$ than those with the first one.

Experimental results with non electrified jets [5-6], show that the Sauter mean diameter $\Theta$ of the droplets produced by disintegration of the jet is proportional to $2\pi/k_{mx}$ and that the initial spray angle $\Omega$ with which droplets are ejected from the jet verifies : $tg\,\Omega\propto\omega_{r_{mx}}/(k_{mx}U_0)$, where $k_{mx}$ is the real wave number corresponding to $\omega_{r_{mx}}$. If we accept these laws for electrified jets, then, the Sauter mean diameter of the droplets diminishes for the equipotential surface case, and slightly increases for the non equipotential one (total equicharged case). With the set of parameters that we used to calculate the results shown on figures 2 and 3, we have a **17.3 %** decrease in the first case and a **1.6 %** increase in the second one. Still considering figures 2 and 3, one can deduce that, this time, the tangent of the initial spray angle increases by **11.3%** in the equipotential case and decreases by **4.3 %** in the second case.

When considering experimental arrangements with finite electrode lengths, as the liquid velocities are high, the time the wave packages spend in the zone where the electric field exists, may be small. However, with high velocities, the maximum of the growing rate is very important and the electric field in a very short time amplifies a wavelength different from the one of the non electrified jet. So, though velocities are high we expect that the electric field will be "felt" by the jet.

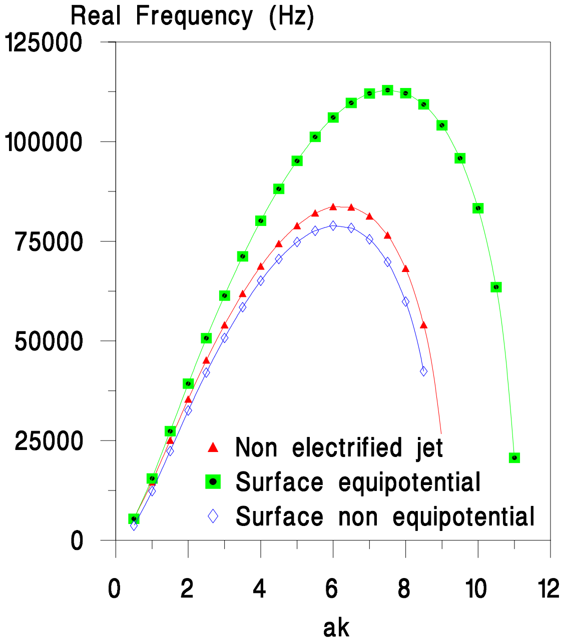


Figure N°2: First mode, Uo=75m/s, a=100µm, a/b=10, surf.tens.=72.2 10-3N/m, E=12.5 MV/m

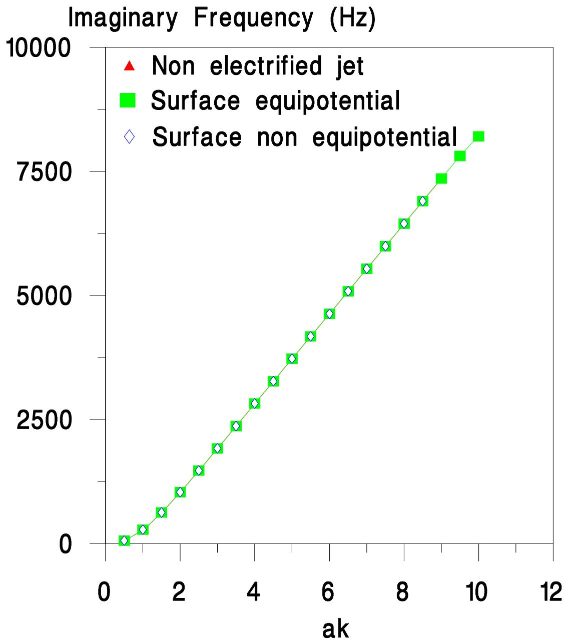


Figure N°3: First Mode, Uo=75m/s, a=100µm, a/b=10, surf.tens.=72.2 10-3N/m, E=12.5MV/m

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