Linear stability analysis of a charged dielectric liquid jet

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Abstract

In this article we develop a linear stability analysis for a dielectric liquid jet with charges on its surface flowing at high speed in a gaseous atmosphere. We analyse the growth rate and the propagation velocity of a perturbation, the jet break-up and the influence of the charges on them.

Introduction

Electric forces applied on moving liquids may change drastically the characteristics of the flow. The electrified jets may be considered as good examples of these phenomena. The changes on these flows have been detected long time ago and one of the first experiences reported are probably the ones of William Gilbert in 1600 [1]. From then until now, different authors have studied the behaviour of the electrified jets and their break-up in droplets.
Most of these works have been concerned with low velocities jets (Rayleigh regime) and very little is known about the changes one can observe in high velocities jets when they are electrified. Recently our laboratory has been working in this very problem [2].
In this article we show some theoretical aspects concerning the changes of the stability of a dielectric liquid jet when electric forces act on its surface looking specially at the case of high speed jets.

Description of the problem

We consider a non perturbed state of an infinite dielectric liquid cylinder inside an earthed coaxial cylindrical electrode. The jet environment is a gaseous atmosphere, initially at rest, and the jet flows at a constant velocity $\vec{U}_0$.
Our system is schematised in figure 1. We describe it considering a cylindrical system of co-ordinates placed in the axis of the jet and moving with velocity $\vec{U}_0$. So the interface is defined by the vector position $\vec{OM} = r_s\ \vec{i}_r + z\ \vec{i}_z$

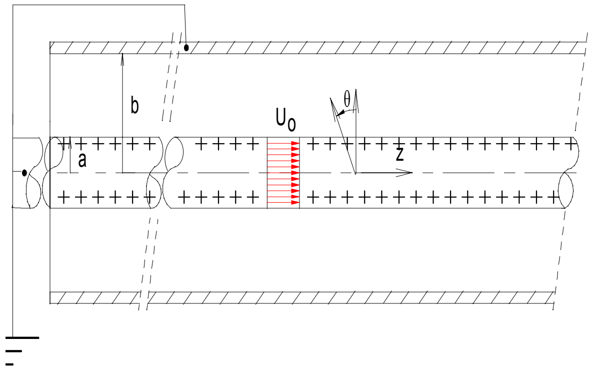

Figure 1

In our study we neglect the viscous effects, the mass transfer between phases and the gravitational and magnetic effects. Both fluids are considered incompressible and the liquid jet velocity is considered uniform in the cross section.
We analyse the case of a jet that has charges uniformly distributed on its surface and that is earthed at infinity, upstream.

General Equations.

We write the classical equations obtained from:

The mass conservation:
- $\vec{\nabla}.\vec{U}_i = 0$ in each phase. $\vec{U}_i$ is the velocity of the phase *i*.(with *i=1* for the liquid and *i=2* for the gas).
- $\vec{\nabla}_{\mathbf{s}}\mathbf{.}\vec{\mathbf{V}}_\zeta = 0$ at the interface. $\vec{V}_\zeta$ is the velocity of the interface and $\vec{\nabla}_s$ the surface nabla.

The absence of mass transfer between the phases:
- $(\vec{U}_1 - \vec{U}_2).\vec{n} = 0$; $(\vec{U}_i - \vec{V}_\zeta).\vec{n} = 0$.

$\vec{n}$ being the unitary vector normal to the interface

The momentum conservation:
- $\rho_i \dfrac{d\vec{U}_i}{dt} - \vec{\nabla} \circ \vec{\vec{T}}_i = \vec{0}$ in each phase. ρ_i and $\vec{\vec{T}}_i$ are the mass density and Maxwell's constraint tensor of

the phase *i*. The expression of this tensor is: $\vec{\vec{T}}_i = -p_i \vec{\vec{I}} + \vec{\vec{T}}_i^{el}$, where p_i is the static pressure and $\vec{\vec{I}}$ the identity tensor [3]. As the fluids are incompressible the components of $\vec{\vec{T}}_i^{el}$, using Einstein's notation, may be written as: $(T_i^{el})_{jk} = \varepsilon_i (E_i)_k (E_i)_j - \frac{\varepsilon_i}{2} \delta_{kj} (E_i)_m (E_i)_m$, with ε_i and $(E_i)_j$ the permittivity and the electric field in the *jth* direction, of the phase *i*.

6) $-div_s(\vec{\vec{T}}_\zeta) - (\vec{\vec{T}}_1 - \vec{\vec{T}}_2) \circ \vec{n} = \vec{\vec{0}}$ at the interface. $\vec{\vec{T}}_\zeta$ is the constraint tensor in this region. It may be expressed as $\vec{\vec{T}}_\zeta = \gamma \ (\vec{\vec{I}} - \vec{n} \otimes \vec{n})$. According to [4] $\vec{\nabla}_s \circ \vec{\vec{T}}_\xi = \gamma < \upsilon_n > \vec{n} + \vec{\nabla}_s \gamma$, being $< \upsilon_n >$ the averaged surface curvature, γ being the surface tension between both phases. As we consider γ constant, the surfacic gradient of this magnitude vanishes.

The Maxwell's equation

We assume that magnetic effects are negligible. So, we can write the expressions of Gauss' and Faraday's law as :

$$\vec{\nabla}\,\vec{D}_i = q_l$$
$$\vec{\nabla} \times \vec{E}_i = \vec{0}$$

both invariant to a Galilean transformation where $\vec{D}_i$ is the electric displacement vector and q_l the free charge density

The basic flow

Velocity fields

As we neglect viscous effects, according to [5], the fluid flows are potential which can be expressed by $\vec{U}_i = \vec{\nabla}\varphi_i$ and $\vec{\nabla}^2\varphi_i = 0$. At $r=r_s$, both phases must satisfy $(\vec{U}_i - \vec{V}_\zeta)\,.\vec{n} = 0$ and at the electrode $\vec{U}_2 \,.\vec{i}_r \big|_{r=b} = \mathbf{0}$.

With this boundary conditions, the solution of Laplace equation gives for the velocity fields :

1) In the liquid phase

$$\vec{U}_{\mathbf{1}} = \vec{0}.$$

2) In the gaseous phase

$$\vec{U}_2 = -U_o \ \vec{i}_z$$

Electric fields

Under our hypotheses, we can obtain the electric field in both fluids using Maxwell's equations [3], which give $\vec{E}_i = -\vec{\nabla}\phi_i$, $\vec{\nabla}^2\phi_i = 0$.

The boundary conditions of the electric potential function at the interface are $\left(\varepsilon_2 \vec{E}_2 - \varepsilon_1 \vec{E}_1\right)\vec{n} = \sigma$ and to the continuity of the tangential electric field. It must be also observed that at the electrode: $\phi_2(b) = 0$.

The geometry of our system enables to consider that the electric field lines are mainly in the radial direction and that they are negligible in the axial direction. So the expressions of the electric fields are

1-In the liquid phase

$$\vec{E}_1 = \vec{0}$$

2-In the gaseous phase

$$\vec{E}_2 = -\frac{\sigma_0 \ a}{\varepsilon_2 \ r}$$

Linear Perturbation Analysis

Now we study the modifications of the basic flow due to an arbitrary small disturbance of the shape of the interface. The Laplace transform of the system with respect to time, a Fourier series with respect to θ and a Fourier transform with respect to z may be taken to express the perturbation in the form

$$A(z,r,\theta,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \int d\omega \eta_0(r,k,n,\omega) e^{\omega t + i(kz+n\theta)}$$

where n is an integer and the contour of the ω-integral is the Bromwich contour for the inversion of the Laplace transform.

As we consider a linear analysis, we can solve the general equations of our system after the arbitrary disturbance had occurred, considering independent normal modes each satisfying the linear system.

Consequently, the radial co-ordinate of the disturbed surface for each mode treated separately is: $r_s = a + \eta = a + \eta_o \exp(i(kz + n\theta) + \omega\, t)$. In this article we use the convention of Drazin [6] and we drop the explicit mention of taking only the real part of the perturbation and of the solutions of equations.

After the perturbation the unitary vector normal to the interface $\vec{n}$ and the average surface curvature $\langle \upsilon_n \rangle$ are :

$$\vec{n} = \vec{i}_r - i\frac{n\eta}{a}\vec{i}_\theta - ik\eta\,\vec{i}_z$$

$$< \upsilon_n >= \vec{\nabla}.\vec{n} = \frac{1}{r_s} - \frac{1}{r_s^2}\frac{\partial^2\eta}{\partial\theta^2} - \frac{\partial^2\eta}{\partial z^2}.$$

Since we develop a linear analysis, we neglect all the terms involving η to a power greater than unity.

Velocity fields after the perturbation

Considering now the disturbed state, the velocity field which satisfies Laplace equation with the above boundary conditions is :

1) In the liquid phase

$$\vec{U}_1 = \frac{\omega}{k\, I'_n(ka)}\vec{\nabla}(I_n(kr)\eta).$$

2) In the gaseous phase

$$\vec{U}_2 = -U_o\,\vec{i}_z + \frac{(\omega - iU_o k)}{k}\vec{\nabla}\left(\left(\frac{I_n(kr)}{I'_n(ka)(1-\lambda)} + \frac{K_n(kr)}{K'_n(ka)(1-\frac{1}{\lambda})}\right)\eta\right)$$

in which $\lambda = \frac{I'_n(kb)\,K'_n(ka)}{I'_n(ka)\,K'_n(kb)}$ and I_n and K_n are the modified Bessel functions of first and second kind, I'_n and K'_n are the first derivatives of these Bessel functions with respect to the argument.

Electric fields after the perturbation

As a result of the perturbation the electric field changes. Then, assuming that charges remain at the surface of the jet we can propose the following expression for the surface charge density σ of the perturbed jet

$$\sigma = \sigma_0\, a / r_s$$

σ_0 being the one of the non perturbed state. In a linear analysis. This relationship establishes that the initial total charge remains constant after the perturbation and that charges do not "swim", as they move following the gravity center of the fluid particles of which they are initially surrounded. This condition is usually verified if the electrical relaxation time is large compared to dynamical times of interest.

The solution of Laplace equation with the above boundary conditions gives the following expression for the electric field:

1) In the liquid phase

$$\vec{E}_1 = -\vec{\nabla}\left(\frac{\sigma_0 In(kr)}{\varepsilon_0 In(ka)\left(1 - \frac{\varepsilon_1 I'_n(ka)K_n(ka)}{\varepsilon_0 I_n(ka)K'_n(ka)}\right)}\eta\right)$$

2) In the gaseous phase

$$\vec{E}_2 = \frac{\sigma_0}{\varepsilon_0}\vec{\nabla}\left(a\ln(r/a) - \frac{Kn(kr)}{Kn(ka)\left(\frac{\varepsilon_2}{\varepsilon_1}\psi - 1\right)}\eta\right)$$

with $\psi = \frac{I_n(ka)K'_n(ka)}{I'_n(ka)K_n(ka)}$

Dispersion equation

The substitution of the electric and velocity fields obtained in the perturbed system, inside the equations of momentum conservation, gives a final equation. Considering that the divergence of the electric constrain tensor in a gaseous atmosphere is negligible [3] in a linear analyses we obtain :

$$\frac{\gamma(1-n^2-(ak)^2)}{a^2} - \frac{\rho_1\omega^2 I_n(ak)}{kI'_n(ak)} + \frac{\rho_2\alpha(\omega - iU_o k)^2}{k} - \frac{\sigma_0^2}{\varepsilon_0}\left(\frac{1}{a} + k\varsigma\right) = 0 \quad (1)$$

with $\alpha = \frac{I_n(ka)}{I'_n(ka)(1-\lambda)} + \frac{K_n(ka)}{K'_n(ka)(1-1/\lambda)}$

and $\varsigma = \frac{K'_n(ka)}{K_n(ka)}\frac{1}{(\frac{\varepsilon_2}{\varepsilon_1}\psi - 1)}$. Note that $\varsigma > 0$.

This equation, for axilsymetric mode (*n=0*) and non electrified jets agrees with Levich's one [7].

Kind of analysis

In this article, we study the initial value problem of a perturbation given in the space at *t=0* (temporal analysis) looking for its evolution in time. So, the set of parameters *(ω,k)* of the unstable waves is obtained considering ω complex and *k* real. In this case

$$\omega_r^2 = \frac{k}{a^2(1-\chi)\rho_2\alpha}\left[-\gamma(1-n^2-(ak)^2) + a\frac{\sigma_0^2}{\varepsilon_0}(1+ak\varsigma)\right] - \omega_i^2\chi$$

$$\omega_i = \frac{U_0\,k}{1-\chi}$$

with $\chi = \frac{\rho_1 I_n(ak)}{\rho_2 I'_n(ak)\,\alpha}$. Note that $\chi<0$ as $\alpha<0$.

Propagation of the perturbation

After some time, the amplitude of the mode grows with time if the growth rate ω_r, the real part of ω, is positive or dampens exponentially if it is negative. The group of modes with the higher values of ω_r will soon outstrip the others and dominate the instability.

So the initial disturbance behaves as a wavepacket composed of this group of normal modes (each of them being a single wave) and grows exponentially with their growth rate.

This wavepacket moves at the group velocity

$$V_g = \frac{\partial \omega}{\partial k} \approx -\left.\frac{\partial \omega_i}{\partial k}\right|_{k_{mx}}$$

where $\omega_{i_{mx}}$ is the imaginary part of ω that corresponds to the maximum growth rate $\omega_{r_{mx}}$.

If the velocity of group is not null then the instability is convective and it propagates downstream, growing and spreading in time.

If the group velocity is null then the perturbation is not swept by the flow, but it just spreads from its initial position and grows in time contaminating the whole flow. This kind of instability is named absolute instability.

Obviously if Galilean transform is respected, an instability that is absolute in a certain referential is convective when observed from another.

In our case, for *a<<b* we can write the group velocity as

$$V_g = -\frac{U_0}{1-\chi}\left(1+\frac{ak\,F}{1-\chi}\right)$$

with

$$F = \frac{\rho_1 K'_n(ak)}{\rho_2 K_n(ak)}\left[\left(1-\frac{I_n(ak)I''_n(ak)}{I'^2_n(ak)}\right)-\left(1-\frac{K_n(ak)K''_n(ak)}{K'_n(ak)}\right)\psi\right]$$

and for large values of *ak* it gives

$$V_g = -\frac{U_0}{1+\frac{\rho_1}{\rho_2}}. \qquad (2)$$

Jet break-up

Finally, as a result of the amplification of the perturbation, the jet is broken into droplets. Though the break-up for the Rayleigh regime has been largely analysed (see for instance [8]), the theory that describes this phenomena for high velocity jets yet is not well established. Nevertheless, for non electrified jets some heuristic approach seem to agree correctly with experimental studies [7,9]. They propose that the mean diameter Θ of the droplets produced by the disintegration of the jet is proportional to the wavelength $2\pi/k_{mx}$, that the angle Ω with which droplets are ejected from the jet verifies : $\mathrm{tg}\,\Omega \propto \omega_{r_{mx}} / (k_{mx} U_0)$, and that the break-up length $L_b \propto \frac{1}{\mathrm{tg}\,\Omega}$, with k_{mx} the real wave number corresponding to $\omega_{r_{mx}}$.

In the next point we will assume that this approach can be extended to electrified jets in order to obtain some basic insight.

Discussion

In figures 2 and 3, we show typical results of the growth rate and ω_i as a function of *ak* They are calculated for the two first number mode *n* and the parameter considered are U_0: 50m/s, a: 100µm, b: 1mm, ε_1: 2, γ: 30.0 mN/m and σ_0: 100µC/m^2. The surface charge corresponds to a jet current of 3.15µA and to a specific charge of 2.0 C/m^3.

In the next paragraph we will discuss these results.

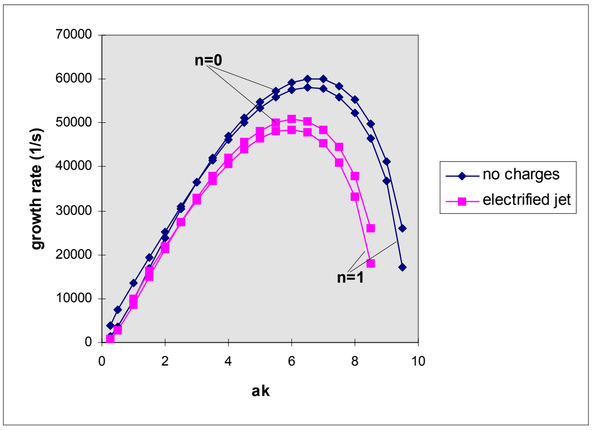

Figure 2

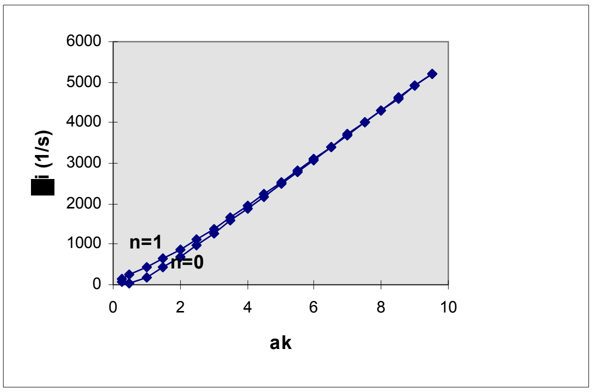

Figure 3: Results of electrified and non electrified jet are superposed

Growth rate of the perturbation

Figures 2 and 3 show that only for a finite continuous interval of *ak*, the equation (1) is satisfied. We observe that, contrary to the Rayleygh regime, for high velocity jets the value of ak_{mx} is greater than 1 and the values of $\omega_{r\,mx}$ are very large.

As we increase the jet velocity, the width of this interval, the values of $\omega_{r\,mx}$ and of ak_{mx} increase.

The results of ω_r we obtain with higher mode numbers n are lesser than those for n=0. As the values of $\omega_{r_{mx}}$ for n=1 are quite close to those for n=0, we can expect that the wavepacket dominating the instability will be composed by axilsymetric and non axilsymetric modes.

For high speed jets, one can deduce from equation (1) that the surface tension and the electric field has a stabilising role (it diminishes ω_r), and that, on the contrary, the effect of the surrounding atmosphere is destabilising (it increases ω_r). The sign of the charge has no effect on the jet stability as in all equations charge density is at a quadratic power.

We can then conclude that as a result of the charges existing on the jet surface the exponentially growth of the disturbance will develop at a lower rate.

However, it must be noted that the estimation of this growth rate is limited by the interference of the different modes of the wave packet that usually leads to an attenuation of this exponential growth of the perturbation. This effect is not considered in this article and our group is working in this very problem now.

Wave propagation

The group velocity of the wavepacket may be directly read from the curves of figure 3 as it is equal to the tangent of this curve calculated at the point $\omega_{i_{mx}}$ time the jet radius a.

We can see that the velocity for the different modes that have the larger ω_r and that compose this wavepacket is almost constant, even if they correspond to different mode number n. So the spread in space of the perturbation, as it grows in time, will be very narrow.

From these curves or from equation (2), we can deduce that in our moving referential for large values of ak the group velocity of the instability is quite close to 0. So the instability observed from a frame fixed to the laboratory is convective and moves with a velocity quite close to the one of the jet.

At low velocities (ak_{mx}<1) the group velocity is smaller and may be modified by the charges on the jet.

However, as it can be observed from figure 3, at jet velocities large enough, the group velocity is independent of the charges on the surface of the jet. The jet when electrified changes slightly the value of k_{mx} but do not change the group velocity $-\left.\frac{\partial \omega_i}{\partial k}\right|_{k_{mx}}$ that is constant in a large interval.

Jet break-up

Considering the approach above mentioned and the data of the figures 2 and 3, when the jet is electrified the mean diameter of the droplets increases in 12.5% the spray angle diminishes 5.1% and the break-up length increases 5.4%.

Conclusions

This paper analyses the stability of a charged dielectric liquid jet. We do not specify the origin of this charges but some phenomena suitable adapted as charging by ion impact of a corona discharge, charge injection inside the nozzle or flow electrification are mechanisms susceptible to produce the configuration we study here.

At high velocity the influence of the charges on the stability of the jet is not very large. However, as the growth rate of the wavepacket is very important these effects should develop very fast in a short axial length.

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